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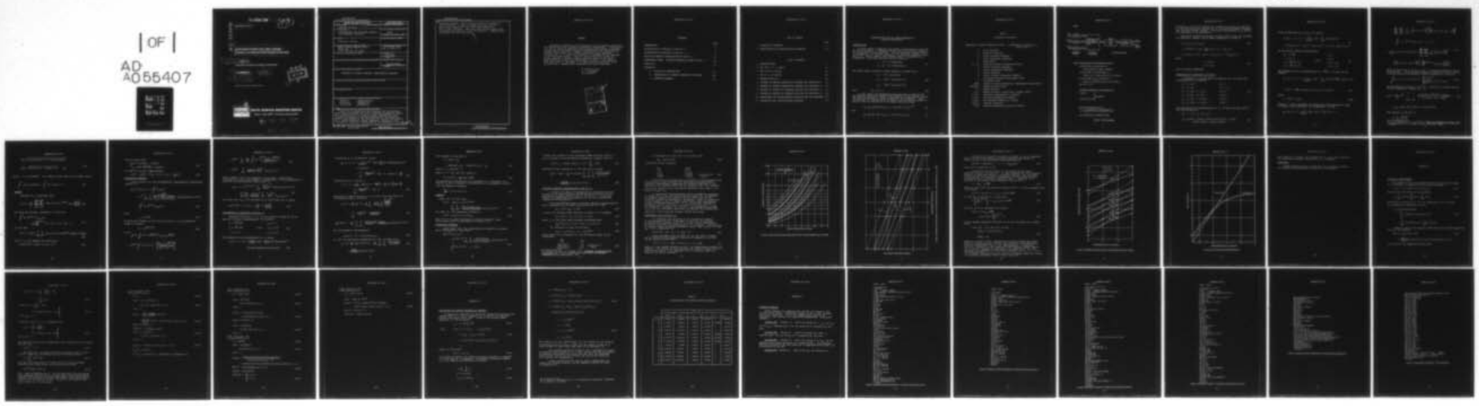

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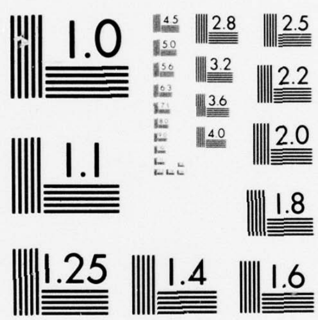

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(10) Leonard E. MILLER

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Theoretical probability distributions for two detector statistics are derived and used to develop performance comparisons. "Statistic A" is a sum of terms representing cross-correlations between a reference channel and two auxiliary channels independent of the reference and of each other except for a common sinusoidal signal component. "Statistic B" is a version of statistic A normalized by the envelope of the

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reference channel. Which of these statistics yields the better detector is shown to depend upon the ratio of noise power received on the auxiliary channels to that on the reference: when this ratio is less than 0.25, the normalized statistic is to be preferred. Computational procedures are fully disclosed.

SUMMARY

Theoretical probability distributions for two detector statistics are derived and used to develop performance comparisons. "Statistic A" is a sum of terms representing cross-correlations between a reference channel and two auxiliary channels independent of the reference and of each other except for a common sinusoidal signal component. "Statistic B" is a version of statistic A normalized by the envelope of the reference channel. Which of these statistics yields the better detector is shown to depend upon the ratio of the noise power received on the auxiliary channels to that on the reference: when this ratio is less than 0.25, the normalized statistic is to be preferred. Computational procedures are fully disclosed.

J. A. Faulkner

J. A. FAULKNER
By direction

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DISTRIBUTIONS FOR TWO CROSS-CORRELATION DETECTOR STATISTICS

INTRODUCTION

A three-channel communications system is postulated to have the following elements: a reference channel and two auxiliary channels, all occupying the same bandwidth. Let the reference channel output be designated $x_0(t)$ and the outputs of the auxiliary channels, $x_1(t)$ and $x_2(t)$. For no signal, each channel output is assumed to be independent, zero-mean Gaussian with the variances

$$\begin{aligned} \sigma_0^2 &= N \text{ (reference)} \\ \sigma_1^2 = \sigma_2^2 &= \alpha N \text{ (auxiliaries)} \end{aligned} \tag{1}$$

The signal power received on these channels is taken to be

$$\begin{aligned} S_0 &= \tfrac{1}{2}S^2 \text{ (reference)} \\ S_1 &= \tfrac{1}{2}k_1^2 S^2 \text{ (auxiliary \#1)} \\ S_2 &= \tfrac{1}{2}k_2^2 S^2 \text{ (auxiliary \#2)} \end{aligned} \tag{1a}$$

with
$$k_1^2 + k_2^2 = 1 \tag{1b}$$

In this study, the probability distributions are derived for two detector statistics obtained from samples of the channel outputs, and some calculations of their performance are presented. Given the glossary of notation shown in Table 1 and the detector model of Figure 1, the two statistics to be studied may be written

$$z_A = \frac{1}{2} Z_0 [Z_1^2 \cos^2(\phi_1 - \phi_0) + Z_2^2 \cos^2(\phi_2 - \phi_0)]^{1/2} \tag{2}$$

and

$$z_B = \frac{1}{2} [Z_1^2 \cos^2(\phi_1 - \phi_0) + Z_2^2 \cos^2(\phi_2 - \phi_0)] \tag{3}$$

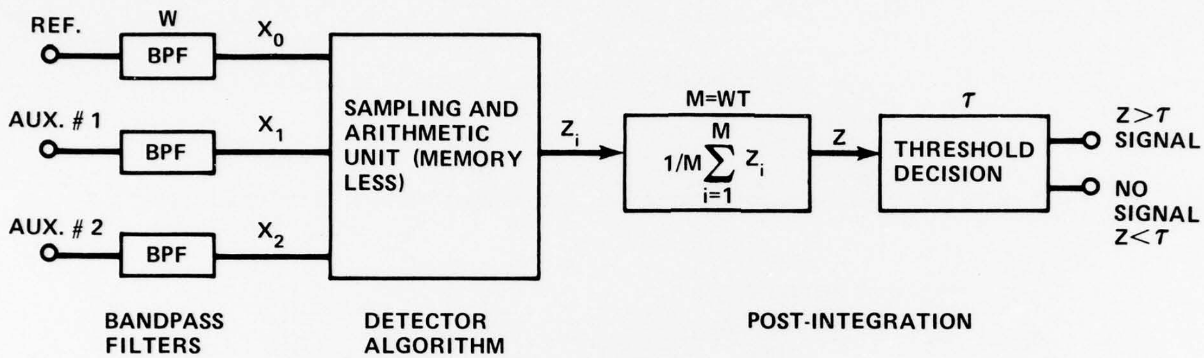
Table 1

Glossary of Notation

Subscripts (unless otherwise noted): 0:reference; 1:auxiliary 1; 2:auxiliary 2

Z	noise envelope
z	detector statistic
a	noise in-phase component
b	noise quadrature component
ϕ	phase angle
k_1, k_2	auxiliary channel weighting factors
ξ	signal in-phase component
η	signal quadrature component
S	signal envelope
N	noise variance (reference channel)
α	auxiliary/reference noise power ratio
τ	threshold value
$M=WT$	number of post-integrations (=bandwidth-time product)
$h^2=S^2/2N$	signal-to-noise ratio
$\Gamma()$	gamma function
$K_n()$	Bessel function, third kind, integer order n
${}_1F_1(;;)$	Confluent hypergeometric function
$Q(x)$	Gaussian probability integral
$Q(x\|v)$	Chi-squared probability integral
$He_n(x)$	Hermite polynomial
$\phi(x)$	Gaussian probability density function

MODEL



INPUTS: GAUSSIAN NOISE PLUS DETERMINISTIC SIGNAL

$$X_0(t) = Z_0(t)\,\mathrm{COS}(\omega_c t - \phi_0) + S(t)\,\mathrm{COS}(\omega_c t - \phi)$$
$$= \left[a_0(t) + \xi(t)\right] \mathrm{COS}\,\omega_c t + \left[b_0(t) + \eta(t)\right] \mathrm{SIN}\,\omega_c t$$
$$X_1(t) = Z_1(t)\,\mathrm{COS}(\omega_c t - \phi_1) + k_1 S(t)\,\mathrm{COS}(\omega_c t - \phi)$$
$$= \left[a_1(t) + k_1\,\xi(t)\right] \mathrm{COS}\,\omega_c t + \left[b_1(t) + k_1\,\eta(t)\right] \mathrm{SIN}\,\omega_c t$$
$$X_2(t) = Z_2(t)\,\mathrm{COS}(\omega_c t - \phi_2) + k_2\,S(t)\,\mathrm{COS}(\omega_c t - \phi)$$
$$= \left[a_2(t) + k_2\,\xi(t)\right] \mathrm{COS}\,\omega_c t + \left[b_2(t) + k_2\,\eta(t)\right] \mathrm{SIN}\,\omega_c t$$

WITH $k_1^2 + k_2^2 = 1$

GAUSSIAN COMPONENTS ARE ZERO-MEAN, WITH

$$\left.\begin{array}{l} E\left\{X_0^2\right\} = N \\ \\ E\left\{X_1^2\right\} = E\left\{X_2^2\right\} = \alpha N \end{array}\right\} \text{NO SIGNAL}$$

DETECTOR STATISTICS (WT=1=M)

$$Z_A = 1/2\sqrt{Z_0^2\,Z_1^2\,\cos^2(\phi_1 - \phi_0) + Z_0^2\,Z_2^2 \cos^2(\phi_2 - \phi_0)}$$

$$Z_B = 1/2\,\left[Z_1^2\,\cos^2(\phi_1 - \phi_0) + Z_2^2 \cos^2(\phi_2 - \phi_0)\right]$$

FIGURE 1 DETECTOR MODEL

Statistic z_B can be interpreted as a normalized version of statistic z_A, and both represent combinations of correlations between reference and auxiliary channel outputs. Successive calculations of z_A and z_B are considered to be independent.

The joint probability density function (pdf) of the six in-phase and quadrature channel output sample components is given by

$$p_1(a_0,b_0,a_1,b_1,a_2,b_2) \tag{4}$$

$$= [\alpha^2(2\pi N)^3]^{-1} \exp\{-\frac{1}{2\alpha N}[\alpha(a_0-\xi)^2+\alpha(b_0-\eta)^2$$

$$+(a_1-\xi k_1)^2+(b_1-\eta k_1)^2+(a_2-\xi k_2)^2+(b_2-\eta k_2)^2]\},$$

where

$$\begin{aligned} \xi &= S\cos\phi \\ \eta &= S\sin\phi \end{aligned} \tag{5}$$

are the signal components.

DISTRIBUTION OF STATISTIC A FOR WT=1

In reference to the pdf shown in Equation (4), we define the transformation of variables:

$$\begin{aligned}
a_0 &= z_0\cos\gamma & z_0 &\geq 0 \\
b_0 &= z_0\sin\gamma & 0 &\leq \gamma \leq 2\pi \\
a_1 &= x_1\cos\gamma - y_1\sin\gamma & -\infty &< x_1 < \infty \\
b_1 &= x_1\sin\gamma + y_1\cos\gamma & -\infty &< y_1 < \infty \\
a_2 &= x_2\cos\gamma - y_2\sin\gamma & -\infty &< x_2 < \infty \\
b_2 &= y_2\sin\gamma + y_2\cos\gamma & -\infty &< y_2 < \infty
\end{aligned} \tag{6}$$

The Jacobean of the transformation is z_0, so that the joint pdf of the new variables is

$$p_2(\ _0, \gamma, x_1, y_1, x_2, y_2) =$$

$$z_0 p_1(z_0\cos\gamma,\ z_0\sin\gamma, x_1\cos\gamma - y_1\sin\gamma, x_1\sin\gamma + y_1\cos\gamma,$$
$$x_2\cos\gamma - y_2\sin\gamma,\ x_2\sin\gamma + y_2\cos\gamma). \tag{7}$$

After integrating out y_1 and y_2 we obtain

$$p_3(Z_0, \gamma, x_1, x_2) = \frac{Z_0}{\alpha(2\pi N)^2} \exp\{-\frac{1}{2\alpha N} [\alpha(Z_0\cos\gamma-\xi)^2$$

$$+ \alpha(Z_0\sin\gamma-\eta)^2 + \{x_1-S k_1 \cos(\gamma-\phi)\}^2 + \{x_2-S k_2 \cos(\gamma-\phi)\}^2]\} . \tag{8}$$

We now define another transformation:

$$\begin{aligned} Z_0 &= v\sqrt{2z} & & z \geq 0 \\ \gamma &= \psi + \phi & & v \geq 0 \\ x_1 &= \sqrt{2z}\cos\zeta/v & \text{with} \quad & 0 \leq \psi \leq 2\pi \\ x_z &= \sqrt{2z}\sin\zeta/v & & 0 \leq \zeta \leq 2\pi \end{aligned} \tag{9}$$

The Jacobean of this transformation is $2\sqrt{2z}/v^2$, so that the pdf becomes

$$p_4(z, \psi, \zeta, v) = \frac{z}{\alpha v(\pi N)^2} \exp\{-\frac{1}{2\alpha N} [\alpha(2zv^2-2\sqrt{2z}Sv\cos\psi + S^2)$$

$$+ 2z/v^2 - 2\sqrt{2z}\, S\cos(\zeta-\theta)\cos\psi/v + S^2\cos^2\psi]\} \tag{10}$$

where

$$\theta = \tan^{-1}(k_2/k_1) \tag{10a}$$

Using $h^2 = S^2/2N$, expanding the second and fifth exponential terms in series, and eliminating v by integration, we find

$$p_5(z, \psi, \zeta) = \frac{z}{\alpha(\pi N)^2} \exp\{-h^2(1 + \cos^2\psi/\alpha)\} \times$$

$$\times \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{m!n!} \left[2h \cos \psi \sqrt{\frac{z}{N}}\right]^m \left[\frac{2h}{\alpha} \cos (\zeta-\theta) \cos \psi \sqrt{\frac{z}{N}}\right]^n$$

$$\times \int_0^{\infty} dv\, v^{m-n-1} \exp \{-\mu_1 v^2 - \mu_2 v^{-2}\}$$

$$= \frac{z}{\alpha(\pi N)^2} \exp \{-h^2 (1+\cos^2\psi/\alpha)\} \sum_{m=0}^{\infty} \sum_{n=o}^{\infty} \frac{1}{m!n!} \left[2h\cos\psi\sqrt{\frac{z}{N}}\right]^m \tag{11}$$

$$\times \left[\frac{2h}{\alpha} \cos(\zeta-\theta) \cos\psi \sqrt{\frac{z}{N}}\right]^n \left(\frac{1}{\alpha}\right)^{(m-n)/4} K_{(m-n)/2} \left[\frac{2z}{N\sqrt{\alpha}}\right],$$

where we have used $\mu_1 = \frac{z}{N}$ and $\mu_2 = \frac{z}{\alpha N}$ in integration formula 3,471.9 of reference 1. After writing the factor $\exp\{-h^2\cos^2\psi/\alpha\}$ in series form and using the integration formula [reference 1, #3,661.2]

$$\int_0^{2\pi} du\, (a \sin u + b \cos u)^k = \begin{cases} 0, & k=2r+1 \\ \frac{2\pi}{2^r}\binom{2r}{r} (a^2+b^2)^r, & k=2r, \end{cases} \tag{12}$$

we successively eliminate ζ and ψ by inegration to obtain the pdf (after recognizing a series for ${}_1F_1$)

$$p_6(z) = \frac{4z}{\alpha N^2} e^{-h^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(h^2 z/N)^{m+n}\, \alpha^{-(m+3n)/2}\, \Gamma(m+n+\frac{1}{2})}{m!\, (m+n)!\, (n!)^2\, \Gamma(m+\frac{1}{2})} \tag{13}$$

$$\times K_{m-n} (2z/N\sqrt{\alpha})\, {}_1F_1 (m+n+\tfrac{1}{2}; m+n+1; -h^2/\alpha).$$

The argument of the pdf is

$$z = \frac{1}{2} Z_0 \sqrt{x_1^2+x_2^2}$$

[1] I. S. Gradshteyn and I. W. Ryzhik, Table of Integrals, Series, and Products (4th Ed.), Academic Press, New York, 1965.

$$= \frac{1}{2} Z_0 \sqrt{(a_1 \cos\gamma + b_1 \sin\ )^2 + (a_2 \cos\gamma + b_2 \sin\gamma)^2}$$

$$= \frac{1}{2} Z_0 \sqrt{Z_1^2 \cos^2(\gamma-\phi_1) + Z_2^2 \cos^2(\gamma-\phi_2)} \quad \equiv Z_A \tag{14}$$

where $\gamma \equiv \phi_0$ and M=WT=1. As a check, we note that for no signal (h^2=0),

$$\int_0^\infty dz\ p_6\ (z;h^2=0) = \int_0^\infty dx\ x\ K_0(x) = 1. \tag{15}$$

MOMENTS

Writing (13) in shortened form,

$$p_6(z) = \frac{4}{\alpha N^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f(m,n;\alpha,h^2,N)\ z^{m+n+1}\ K_{m-n}\left(\frac{2z}{N\sqrt{\alpha}}\right), \tag{16}$$

and using the integral [reference 1, #6.561.16]

$$\int_0^\infty dx\ x^{m+n+\mu+1}\ K_{m-n}(ax)$$
$$= \frac{1}{a^2}\left(\frac{2}{a}\right)^{m+n+\mu} \Gamma(m + \frac{\mu}{2} + 1)\Gamma(n + \frac{\mu}{2} + 1), \tag{17}$$

we find that

$$E\{z^\mu\} = (N\sqrt{\alpha})^\mu \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f(m,n;\alpha,h^2,N)\ (N\sqrt{\alpha})^{m+n}\Gamma(m+\frac{\mu}{2}+1)\Gamma(n+\frac{\mu}{2}+1). \tag{18}$$

For h^2 = 0, the moments are, explicitly,

$$E\{z^\mu|h^2=0\} = (N\sqrt{\alpha})^\mu\ [\Gamma(\ \frac{\mu}{2} + 1)]^2. \tag{19}$$

Thus for noise only,

$$\begin{cases} E\{z\} = N\pi\sqrt{\alpha}/4 = .7854N\sqrt{\alpha} \\ \sigma_z = N\sqrt{\alpha}\,\sqrt{1-\pi^2/16} = .6190N\sqrt{\alpha}. \end{cases} \tag{20}$$

For small h^2 we have, approximately,

$$E\{z^\mu\} \simeq (N\sqrt{\alpha})^\mu \left[\Gamma\left(\frac{\mu}{2}+1\right)\right]^2 \left[1 + \frac{\mu}{2}\left(1+\frac{1}{2\alpha}\right)h^2\right]. \tag{21}$$

PROBABILITY INTEGRAL

Given the pdf (13), the corresponding (complementary) probability integral is

$$Q_A(\tau) \triangleq P_r\{Z \geq \tau\} = \int_\tau^\infty dz\; p_6(z)$$

$$= e^{-h^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\tfrac{1}{2}h^2)^{m+n}\left(\frac{1}{\alpha}\right)^n \Gamma(m+n+\tfrac{1}{2})}{m!\,(n!)^2\,(m+n)!\,\Gamma(m+\tfrac{1}{2})}\; {}_1F_1(m+n+\tfrac{1}{2};m+n+1;-h^2/\alpha)$$

$$\times \int_{\tau_1}^\infty dx\; x^{m+n+1}\, K_{m-n}(x), \tag{22}$$

where

$$\tau_1 = 2\tau/N\sqrt{\alpha}\;. \tag{23}$$

To solve the integral, we use the fact that $K_r = K_{-r}$ and make the change of variable

$$x = \tau_1\sqrt{w^2+1} \tag{24}$$

to get

$$\tau_1^{m+n+1} \int_0^\infty dw\; w\; (\sqrt{w^2+1})^{m+n}\; K_{n-m}(\tau_1\sqrt{w^2+1})$$

$$= \tau_1^{m+n+1} \int_0^\infty dw\; (w^2+1)^n \left\{\frac{wK_{n-m}(\tau_1\sqrt{w^2+1})}{(\sqrt{w^2+1})^{n-m}}\right\}$$

$$= \tau_1^{m+n+1} \sum_{k=0}^{n} \binom{n}{k} \int_0^\infty dw \; \frac{w^{2k+1} K_{n-m}(\tau_1 \sqrt{w^2+1})}{(\sqrt{w^2+1})^{n-m}} \tag{25}$$

$$= \tau_1^{m+n+1} \sum_{k=0}^{n} \left(\frac{n!}{(n-k)!}\right) \left(\frac{2}{\tau_1}\right)^k K_{n-m-k-1}(\tau_1),$$

where integral 6.659.3 of reference (1) was used. Substituting this result in (22), using Kummer's transformation on the hypergeometric function , and manipulating indices gives us

$$Q_A(\tau) = \tau_1 e^{-h^2(1+1/\alpha)} \sum_{m=0}^{\infty} \frac{(\frac{1}{2}h^2\tau_1)^m}{(m!)^2} \Gamma(m+\tfrac{1}{2}) \, {}_1F_1(\tfrac{1}{2};m+1;h^2/\alpha)$$

$$\times \sum_{n=0}^{m} \binom{m}{n} \frac{(2/\alpha\tau_1)^n}{\Gamma(m-n+\frac{1}{2})} \sum_{k=0}^{n} \frac{(\tau_1/2)^k}{k!} K_{m-n-k+1}(\tau_1). \tag{26}$$

For noise only, Q_A is the probability of false alarm and is simply

$$Q_A(\tau|h^2=0) = \tau_1 K_1(\tau_1) = \frac{2\tau}{N\sqrt{\alpha}} \; K_1\left(\frac{2\tau}{N\sqrt{\alpha}}\right). \tag{27}$$

DISTRIBUTION OF STATISTIC B FOR WT = 1

Beginning with $p_3(z_0, \gamma, x_1, x_2)$ as written in equation (8), we make the following transformation of variables:

$$\begin{aligned} x_1 &= \sqrt{2\lambda}\cos\delta & & 0 \le \lambda < \infty \\ x_2 &= \sqrt{2\lambda}\sin\delta & \text{with} \quad & 0 \le \delta \le 2\pi \\ \gamma &= \psi + \phi & & 0 \le \psi \le 2\pi \end{aligned} \tag{28}$$

The Jacobean of the transformation is 1, giving

$$p_7(Z_0, \lambda, \delta, \psi) = \frac{z_0}{\alpha(2\pi N)^2} \exp\{-\frac{1}{2\alpha N}[\alpha(Z_0^2 - 2Z_0 S\cos\psi + S^2)$$

$$+2\lambda - 2S\sqrt{2\lambda}\cos\psi\cos(\delta-\theta) + S^2\cos^2\psi]\} \tag{29}$$

Eliminating Z_0 by integration, we get

$$p_8(\lambda, \delta, \psi,) = \frac{1}{\alpha N(2\pi)^2} e^{-h^2} \exp\{ \frac{-1}{2\alpha N}[2\lambda - 2S\sqrt{2\lambda}\cos\psi\cos(\delta-\theta) + S^2\cos^2\psi]\}$$

$$\times \sum_{m=0}^{\infty} \frac{(2h\cos\psi)^m}{m!} \Gamma(\frac{m}{2}+1), \text{ using } h^2 = \frac{S^2}{2N}. \tag{30}$$

Integrating over δ results in

$$p_9(\lambda, \psi) = \frac{1}{2\alpha N\pi} \exp\{-h^2(1+\frac{\cos^2\psi}{\alpha}) - \frac{\lambda}{\alpha N}\} I_0(\frac{2h}{\alpha}\cos\psi\sqrt{\frac{\lambda}{N}})$$

$$\times \sum_{m=0}^{\infty} \frac{(2h\cos\psi)^m}{m!} \Gamma(\frac{m}{2}+1) \tag{31}$$

Expanding the Bessel function in its series form and using (12) to integrate over ψ, we obtain

$$p_{10}(\lambda) = \frac{1}{\alpha N} \exp\{-h^2 - \frac{\lambda}{\alpha N}\} \sum_{m=0}^{\infty} \frac{h^{2m}}{\Gamma(m+\frac{1}{2})} \sum_{n=0}^{\infty} \frac{\left(\frac{h}{\alpha}\sqrt{\frac{\lambda}{N}}\right)^{2n}}{n!n!}$$

$$\times \sum_{k=0}^{\infty} \frac{(-h^2/\alpha)^k}{k!} \frac{\Gamma(m+n+k+\frac{1}{2})}{(m+n+k)!} \tag{32}$$

$$= \frac{1}{\alpha N} \exp\{-h^2 - \frac{\lambda}{\alpha N}\} \sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{h^{2m}(\lambda/\alpha^2 N)^n}{m!(n!)^2} \frac{\Gamma(m+\frac{1}{2})}{\Gamma(m-n+\frac{1}{2})} {}_1F_1(m+\tfrac{1}{2}; m+1; -h^2/\alpha).$$

By using Kummer's transformation

$${}_1F_1(a; b; -c) = e^{-c} {}_1F_1(b-a; b; c), \tag{33}$$

on (32), an alternative expression for (32) is found to be

$$p_{10}(\lambda) = \frac{1}{\alpha N} \exp\{-h^2(1+1/\alpha) - \frac{\lambda}{\alpha N}\} \sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{h^{2m}(\lambda/N\alpha^2)^n}{m!\,(n!)^2} \tag{34}$$

$$\times \frac{\Gamma(m+\frac{1}{2})}{\Gamma(m-n+\frac{1}{2})} {}_1F_1(\tfrac{1}{2}; m+1; h^2/\alpha).$$

The argument of this pdf is

$$\lambda = \tfrac{1}{2}(x_1^2 + x_2^2)$$

$$= \tfrac{1}{2}[Z_1^2\cos^2(\gamma-\phi_1) + Z_2^2\cos^2(\gamma-\phi_2)] \equiv z_B \tag{35}$$

where $\gamma \equiv \phi_o$ and WT = 1.

When $h^2 = 0$, (32) and (34) reduce to

$$p_{10}(\lambda;h^2=0) = \frac{1}{\alpha N}\exp\{-\lambda/\alpha N\}, \tag{36}$$

the exponential distribution with parameter αN. Therefore, to a factor, z_B (for no signal) is distributed as a chi-squared variable with two degrees of freedom:

$$\frac{2z_B}{\alpha N} \sim \chi^2(2) \text{ for } h^2 = 0. \tag{37}$$

MOMENTS

Using (34), we find that

$$E\{\lambda^\mu\} = (N\alpha)^\mu\, \mu!\, e^{-h^2(1+1/\alpha)} \times \sum_{m=0}^{\infty}\sum_{n=0}^{m} \frac{h^{2m}\alpha^{-n}\Gamma(m+\frac{1}{2})(\mu+1)_n}{m!(n!)^2\Gamma(m-n+\frac{1}{2})}\, {}_1F_1(\tfrac{1}{2};m+1;h^2/\alpha). \tag{38}$$

For small h^2, this expression reduces to

$$E\{\lambda^\mu\} \cong (N\alpha)^\mu\, \mu!\,(1+h^2/2\alpha); \tag{39}$$

thus for $h^2 = 0$, both the mean and standard deviation equal $N\alpha$--a result which is predictable in view of (37).

PROBABILITY INTEGRAL

Again using (34), the (complementary) probability integral which corresponds is found from

$$Q_B(\tau) \triangleq P_r\{\Lambda \geq \tau\}$$

$$= e^{-h^2(1+1/\alpha)} \sum_{m=0}^{\infty}\sum_{n=0}^{m} \frac{h^{2m}\alpha^{-n}\Gamma(m+\frac{1}{2})}{m!(n!)^2\Gamma(m-n+\frac{1}{2})}\, {}_1F_1(\tfrac{1}{2};m+1;h^2/\alpha) \times \int_{\tau_1}^{\infty} du\, u^n e^{-u}, \quad \tau_1 = \tau/\alpha N, \tag{40}$$

in which the integral is the incomplete gamma function $\Gamma(n+1;\tau_1)$, and is related to the chi-squared probability integral $Q(x^2|\nu)$:

$$\Gamma(n+1;\tau_1) = n!Q(2\tau_1|2n+2) = n!\ e^{-\tau_1} \sum_{k=0}^{n} \tau_1^k/k! \tag{41}$$

Putting the last expression in (41) into (40), we have

$$Q_B(\tau) = \exp\{-\frac{\tau}{\alpha N} - h^2(1+1/\alpha)\} \sum_{m=0}^{\infty} \sum_{n=0}^{m} \sum_{k=0}^{n} \frac{h^{2m}\alpha^{-n}(\tau/\alpha N)^k}{m!\ n!\ k!}$$

$$\times \frac{\Gamma(m+\frac{1}{2})}{\Gamma(m-n+\frac{1}{2})}\ {}_1F_1(\tfrac{1}{2};\ m+1;\ h^2/\alpha). \tag{42}$$

RECEIVER OPERATING CHARACTERISTICS FOR WT = 1

In order to compare the properties of statistics A and B, we shall first obtain receiver operating characterstics from the expressions already derived, that is, for WT = 1. In the next section, an approximation technique will be used to extend the comparison to arbitrary WT.

The relationships known as receiver operating characteristics may be expressed in the present application by (h^2 = SNR)

$$P_D = f(h^2;\ P_{FA},\ \alpha,\ WT) \tag{43}$$

in which the decision model depicted in Figure (1) is assumed:

$$P_D = \Pr\{z > \tau_a\} = Q_i(\tau_a),\ i = A,B \tag{44}$$

where τ_a is the false alarm threshold determined from

$$P_{FA} \triangleq Q_i(\tau_a;\ h^2 = 0) = \Pr\{z > \tau_a;\ h^2 = 0\}. \tag{45}$$

For statistic A, from (27) we have

$$P_{FA} = \tau_{1a}\ K_1(\tau_{1a}),\ \tau_{1a} = 2\tau_a/N\sqrt{\alpha}. \tag{46}$$

Using Table 9.8 of reference (2), the following values can be calculated:

P_{FA}	τ_{1a}		
$.0097 \approx 10^{-2}$	5.8	(Statistic A)	
$.00103 \approx 10^{-3}$	8.2	WT=1	(47)
$.000105 \approx 10^{-4}$	10.6		

[2] M. Abramowitz and I. A. Stegun, eds., Handbook of Mathematical Functions, NBS Applied Mathematics Series #55, Government Printing Office, Washington, 1970.

For statistic B, from (42) it is evident that

$$P_{FA} = \exp\{-\tau/\alpha N\}, \tag{48}$$

from which we may calculate

P_{FA}	$\tau_a/N\alpha$		
0.01	4.60517		
0.001	6.90776	(Statistic B)	(49)
0.0001	9.21034	WT = 1	

Calculation of the probability integrals to find probabilities of detection is more involved, but relatively straightforward. Using the computational approaches and programs described in the appendices, Q_A (equation (26)) and Q_B (equation (42)) were obtained for several values of the auxiliary-to-reference power ratio α, as shown in Figure (2).

As expected an obvious feature of the information displayed in Figure (2) is that, for fixed reference power (N), performance is improved (smaller SNR required) as the noise power on the auxiliary channels is decreased (α decreased). What is interesting in this figure is what it reveals about the relative performance of the two detector statistics. It is evident that, because of different sensitivities to α, statistic A is better for $\alpha > 0.25$ but statistic B is better for $\alpha \leq 0.25$. The interpretation seems to be that the reference's amplitude information (Z_o) becomes more important as the auxiliary channels become more noisy.

PERFORMANCE STUDY FOR WT > 1

To illustrate the extension of the theoretical results we have obtained for WT = 1 to cases in which WT is arbitrary, we examine the crossover effect of α on the relative performance of the two statistics for consistency as WT increases. Specifically, we compare values of minimum detectable signal (MDS) for $P_D = 0.5$ and $P_{FA} = 0.01$, expressible as

$$\text{MDS} \triangleq h^2\{\alpha,\ WT:\ P_D = .5,\ P_{FA} = .01\} \tag{50}$$

Using the moments for WT shown in (18) and (38) as inputs to the Cornish-Fisher expansion described in Appendix B, we find MDS by varying the SNR until

$$\tau(h^2;\ Q = .5,\ \alpha,\ WT) = \tau(h^2 = 0;\ Q = .01,\ \alpha, WT) \tag{51}$$

where τ is the inverse function of $Q(\tau)$, the complementary probability integral. The results of this procedure are shown in Figure (3), in which we see that $\alpha = .25$ continues to be a crossover value for choosing the better statistic.

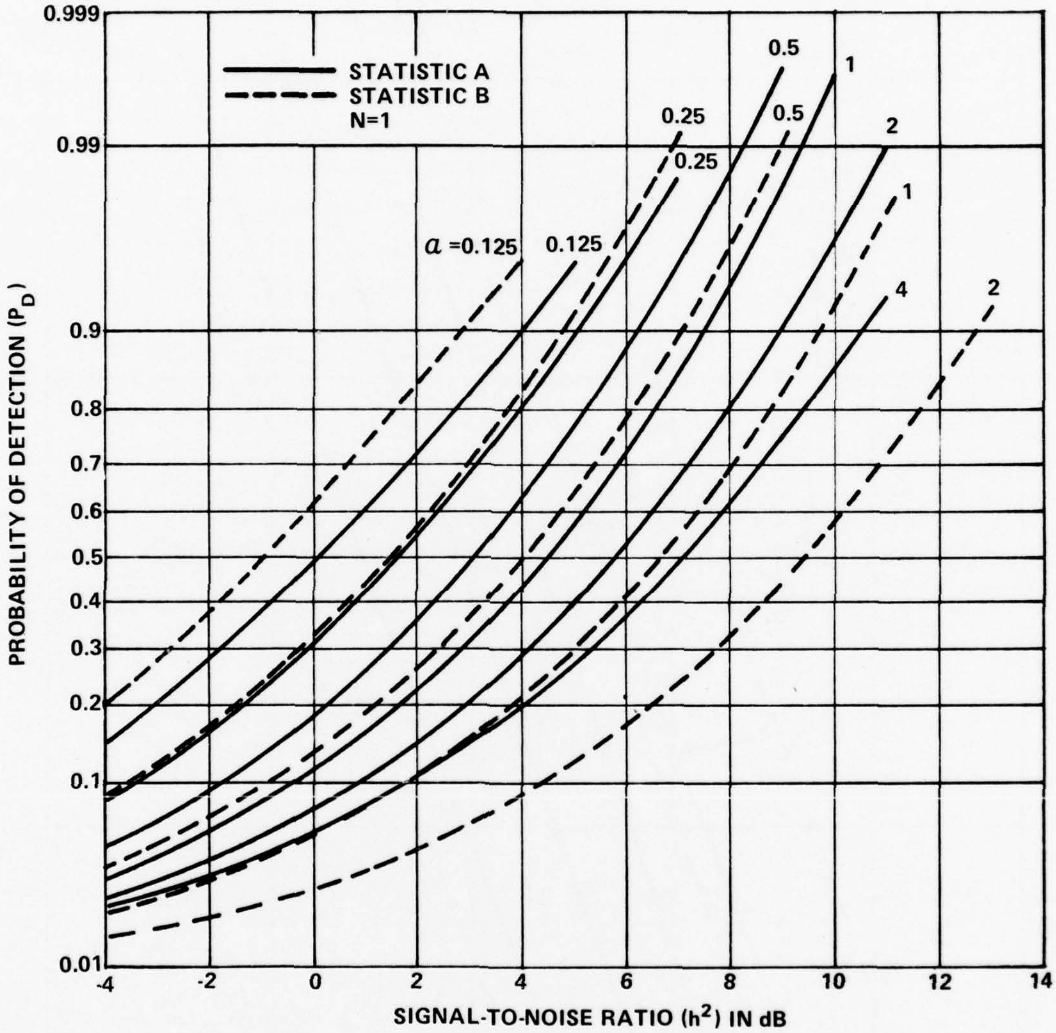


FIGURE 2 RECEIVER OPERATING CHARACTERISTICS (WT=1), NOISE POWER RATIO (a) VARIED

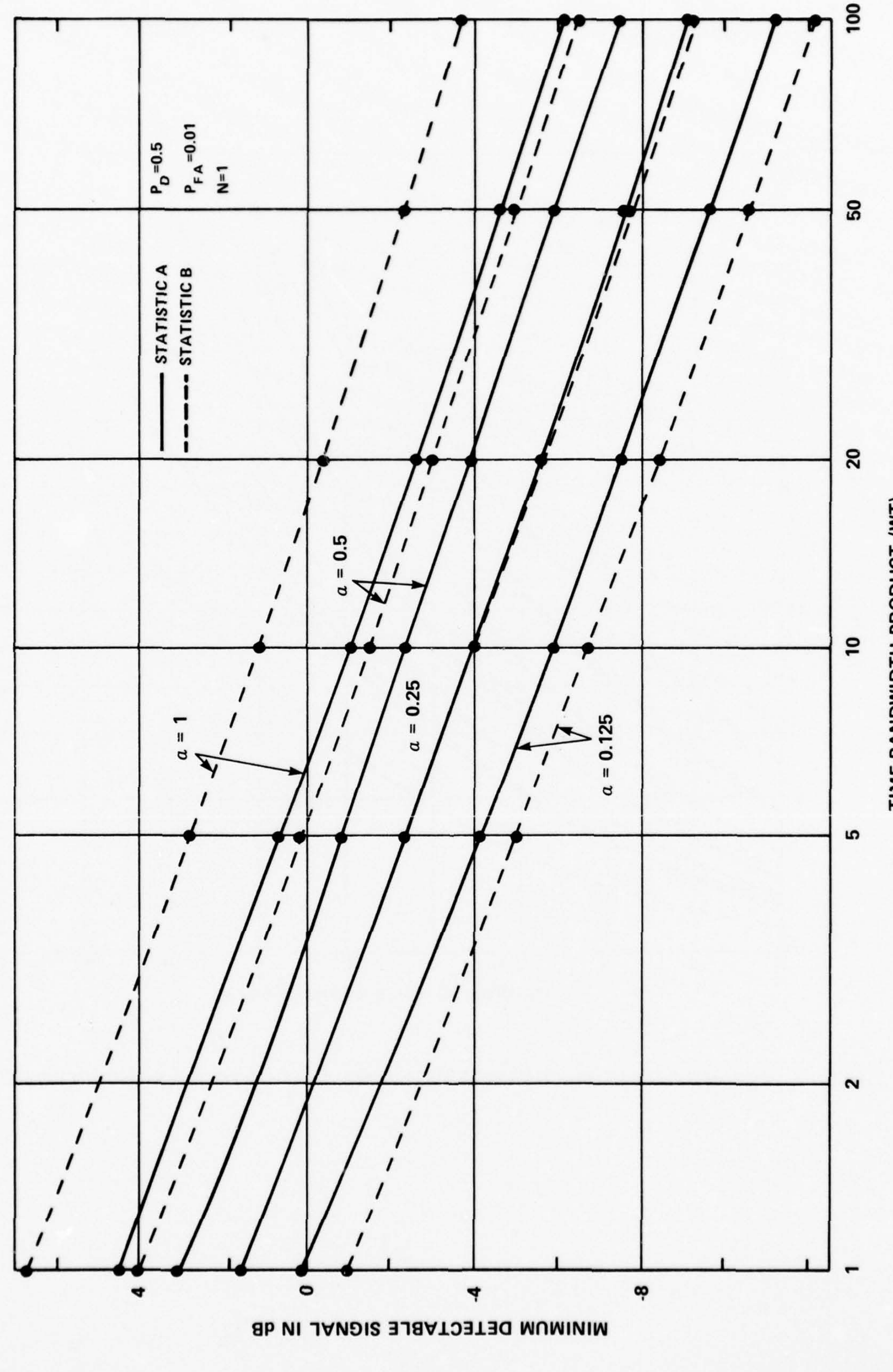


FIGURE 3 MINIMUM DETECTABLE SIGNAL VS TIME-BANDWIDTH PRODUCT, NOISE POWER RATIO (a) VARIED

The behavior of the MDS as shown in Figure (3) is so consistent that it gives us confidence in further extending the results of Figure (2) with such approximate generalizations as

$$MDS(WT) = MDS(WT=1) - .5 - 5 \log_{10}(WT), \tag{52}$$

where MDS is specified in dB.

Taking the data of Figure (3) and making WT the family parameter results in Figure (4). In this presentation, the crossover behavior we have been noting is displayed directly. The curvature of the results for statistic A discourages us from making any bold statements about the general dependence upon α. However, if we make use of the central limit theorem to say, as $WT \to \infty$.

$$z \sim N(m_z, \sigma_z/\sqrt{WT}) \tag{53}$$

where m_z and σ_z^2 are the mean and variance for WT = 1, then asymptotically (51) becomes

$$m_z(h^2) = m_z(h^2= 0) + x_{.01}\, \sigma_z(h^2=0)/\sqrt{WT}. \tag{54}$$

In addition, since $h^2 << 1$, we may write

$$h^2 \left[\frac{\partial m_z}{\partial h^2} \Big|^{h^2=0} \right] = x_{.01}\, \sigma_z(h =0)/\sqrt{WT}. \tag{55}$$

Finally, if as in reference 3 we define

$$\begin{aligned} \Delta MDS &= h^2/(x_{.01}/\sqrt{WT}) \\ &= \sigma_z(h =0)/[\ \frac{\partial m_z}{\partial h^2}\Big|^{h^2=0}\ ], \end{aligned} \tag{56}$$

we can compare asymptotic behavior of the two statistics on a common basis.

From (20), (21), and (39), we get

$$\begin{aligned} \Delta MDS_A &= 1.576/(1+1/2\alpha) \\ \Delta MDS_B &= 2\alpha. \end{aligned} \tag{57}$$

Figure 5, a plot of (57), confirms what we have already concluded from the previous results, namely, that statistic A provides better detection performance for $\alpha > .25$, approximately. In addition, for $\alpha >> 1$, statistic A's MDS relative to the baseline (which happens to be that of a square law statistic) is no worse than 2 dB. We see also, that although B is the better detector for $\alpha \leq .25$, A is at

[3] C. N. Pryor, "Calculation of the Minimum Detectable Signal for Practical Spectrum Analyzers," NOLTR 71-92, 2 Aug 1971.

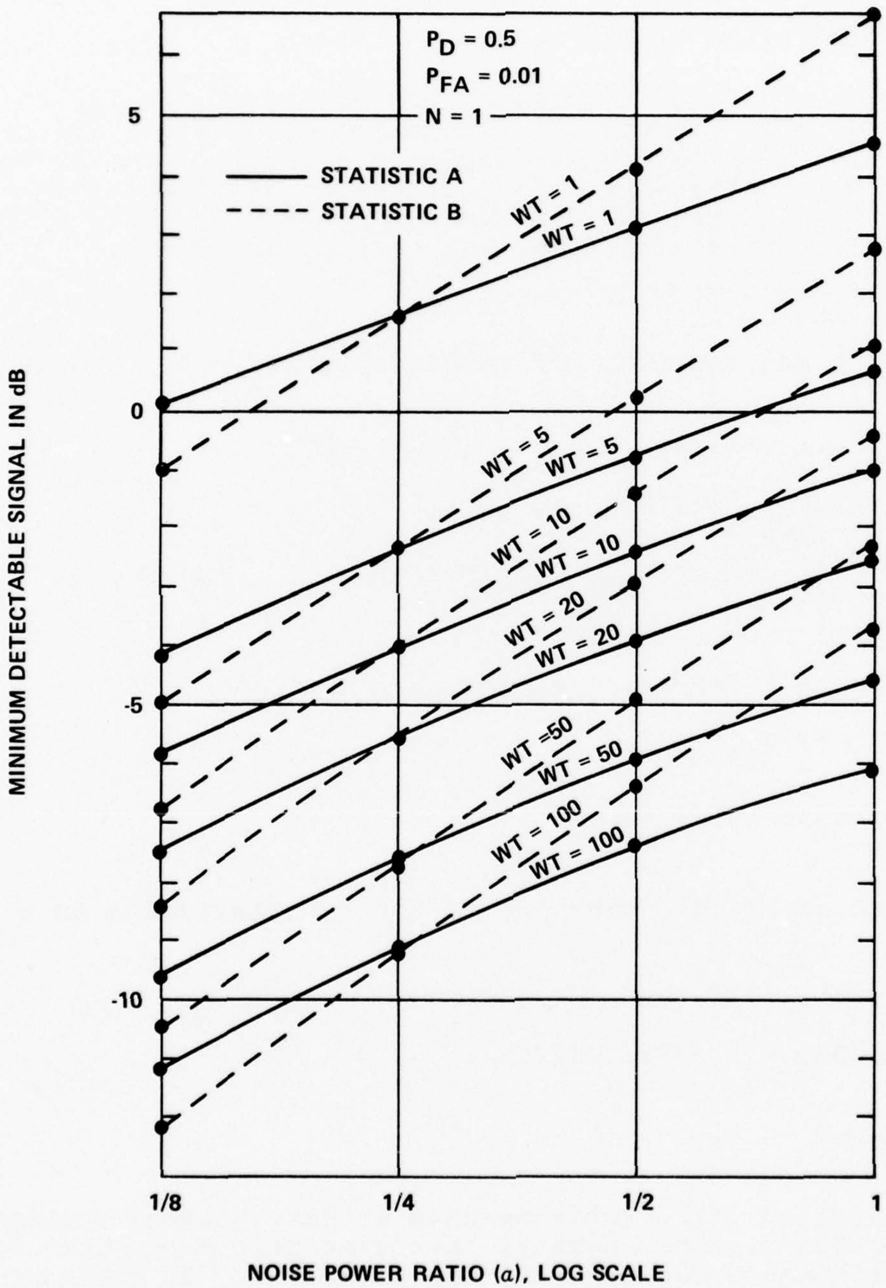


FIGURE 4 MINIMUM DETECTABLE SIGNAL VS NOISE POWER RATIO,WT VARIED

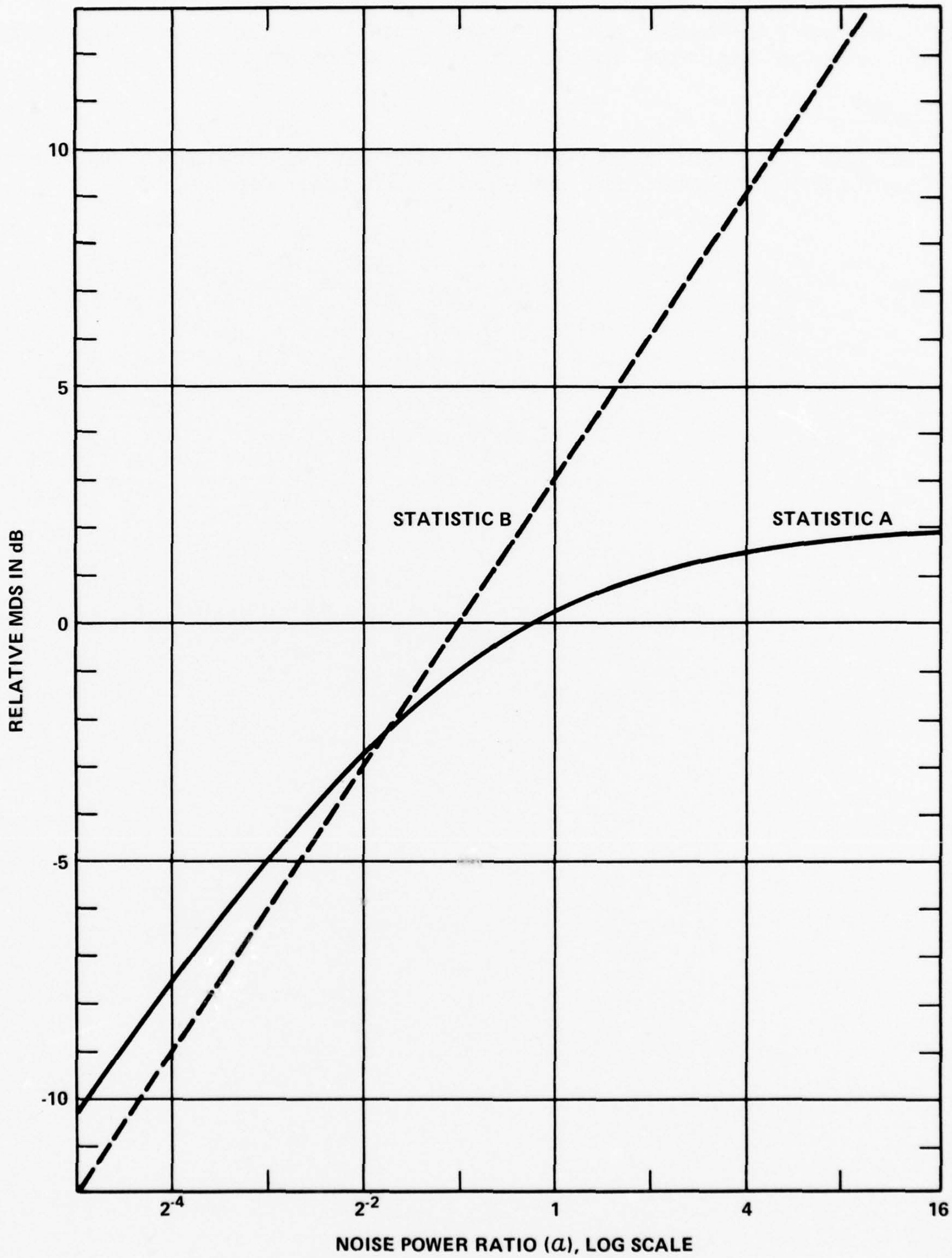


FIGURE 5 RELATIVE MDS VS NOISE POWER RATIO

most worse by 2 dB here, too; whereas when B is the worse detector, its performance degrades indefinitely as α increases.

CONCLUSION

Further comparisons can be made with the theoretical results and computational procedures documented in this report.

APPENDIX A

OUTLINE OF COMPUTATIONS

The general form of the computations which are special enough to be discussed in themselves is the infinite series

$$f_1(x) = f_2(x) \exp\{-h^2 - h^2/\alpha\} S_2(x) \tag{A-1}$$

where, given the values of the arguments,

$$S_2 = \sum_{m=0}^{\infty} A(m)G(m) \sum_{n=0}^{m} B(m, n) \sum_{k=0}^{n} C(k). \tag{A-2}$$

Truncating the series at $m = m_\infty$ and expanding on (A-2) to show the computational approach we write

$$\begin{aligned} S_2 = {} & A(0)G(0)B(0,0)C(0) \\ & + \sum_{m=1}^{m_\infty} A(m)G(m) \{B(m,0)C(m,0,0) \\ & + \sum_{n=1}^{m} B(m,n)[C(m,n,0) + \sum_{k=1}^{n} C(m,n,k)]\} \end{aligned} \tag{A-3}$$

Common to each of the specific forms which are treated separately below is the function

$$\begin{aligned} G(m) & \equiv {}_1F_1(\tfrac{1}{2};\ m+1;\ h^2/\alpha) \\ & = \frac{2m\alpha}{(2m-1)h^2} [(m-1+h^2/\alpha)\ G(m-1) - (m-1)G(m-2)],\ m \geq 2\ . \end{aligned} \tag{A-4}$$

G(0) and G(1) are computed directly from

$$_1F_1(\tfrac{1}{2};\ 1;\ x) = \sum_{r=0}^{\infty} \frac{x^r}{(r!)^2} (\tfrac{1}{2})_r$$

$$\simeq \sum_{r=0}^{r\infty} D_0(r) \qquad \text{(A-5)}$$

$$\left.\begin{aligned} \text{using } D_0(r) &= \frac{x^r}{(r!)^2} (\tfrac{1}{2})_r \\ &= x(r-\tfrac{1}{2})D_0(r-1)/r^2,\ r \geq 1 \\ \text{with } D_0(0) &= 1. \end{aligned}\right\} \qquad \text{(A-6)}$$

Similarly, $_1F_1(\tfrac{1}{2};\ 2;\ x)$ is computed using

$$\left.\begin{aligned} D_1(r) &= \frac{x^r}{r!} \frac{(\tfrac{1}{2})_r}{(r+1)!} \\ &= x(r - \tfrac{1}{2})D_1(r-1)/r\ (r+1) \\ \text{with } D_1(0) &= 1. \end{aligned}\right\} \qquad \text{(A-7)}$$

Satisfactory results were accomplished when truncation of the series was such that

$$D_0(r_\infty) < 10^{-13}. \qquad \text{(A-8)}$$

The form (A-2), besides being more convenient than a triple infinite summation, was chosen to imitate the power series

$$S_2 = \sum_{m=0}^{m\infty} (h^2)^m\ E(m), \qquad \text{(A-9)}$$

with the idea being that h^2 is most often the major parameter (sometimes also being large) and truncation such that

$$(h^2)^{m_\infty} E(m_\infty) < .0001\ S_2 \qquad \text{(A-10)}$$

will insure convergence in h^2. For the most part this has worked well, although for $\alpha > 2$ and $\alpha < \tfrac{1}{2}$ in some cases a more sophisticated approach (not assuming so uniform convergence) might be warranted, judging from the behavior of $Q_A(\tau)$ and $Q_B(\tau)$, for example, when their values are 0.9 and higher.

$Q_A(\tau)$--Equation (26)

$$f_2 \equiv \tau_1 \quad \text{(A-11)}$$

$$A(m) \equiv (h^2\tau_1/2)^m/(m!)^2$$

$$= (h^2\tau_1/2)\ A(m-1)/m^2,\ m \geq 1 \quad \text{(A-12)}$$

$$A(0) = 1$$

$$B(m,n) \equiv \binom{m}{n} \frac{\Gamma(m+\frac{1}{2})}{\Gamma(m-n+\frac{1}{2})} (2/\alpha\tau_1)^n$$

$$= \left(\frac{2}{\alpha\tau_1}\right) (m-n+1)\ (m-n+\tfrac{1}{2})\ B(m,\ n-1)/n,\ n \geq 1 \quad \text{(A-13)}$$

$$B(m,0) = B(0,0) = 1$$

$$C(m,n,k) \equiv C_1(k) K_{m-n-k+1}(\tau_1)$$

$$C_1(k) = (\tau_1/2)^k/k! \quad \text{(A-14)}$$

$$= (\tau_1/2)\ C_1(k-1)/k,\ k \geq 1$$

$$C_1(0) = 1$$

$$K_{r+1}(\tau_1) = 2rK_r(\tau_1)/\tau_1 + K_{r-1}(\tau_1),\ r \geq 2 \quad \text{(A-15)}$$

$$K_{-r}(\tau_1) = K_r(\tau_1)$$

$e^{\tau_1} K_0(\tau_1)$ and $e^{\tau_1} K_1(\tau_1)$ tabulated in reference (2).

$Q_B(\tau)$--Equation (42)

$$f_2 \equiv \exp(-\tau/\alpha N) \qquad \text{(A-16)}$$

$$A(m) \equiv (h^2)^m/m!$$

$$= (h^2)\ A(m-1)/m,\ m \geq 1 \qquad \text{(A-17)}$$

$$A(0) = 1$$

$$B(m,n) \equiv \Gamma(m+\tfrac{1}{2})/\alpha^n n!\Gamma(m-n+\tfrac{1}{2})$$

$$= (m-n+\tfrac{1}{2})B(m,n-1)/\alpha n,\ n \geq 1 \qquad \text{(A-18)}$$

$$B(m,0) = B(0,0) = 1$$

$$C(k) \equiv (\tau/\alpha N)^k/k!$$

$$= (\tau/\alpha N)\ C(k-1)/k,\ k \geq 1 \qquad \text{(A-19)}$$

$$C(0) = 1$$

$E\{z_A^\mu\}$--Equation (18)

$$f_2 \equiv (N\sqrt{\alpha})^\mu \qquad \text{(A-20)}$$

$$A(m) \equiv (h^2)^m/(m!)^2$$

$$= h^2A(m-1)/m^2,\ m \geq 1 \qquad \text{(A-21)}$$

$$A(0) = 1$$

$$B(m,n) \equiv \frac{\Gamma(m+\tfrac{1}{2})\Gamma(m+1)\Gamma(m-n+\mu/2+1)\Gamma(n+\mu/2+1)}{\Gamma(m-n+\tfrac{1}{2})\Gamma(m-n+1)(n!)^2\ \alpha^n}$$

$$= (m-n+\tfrac{1}{2})(m-n+1)(n+\mu/2)B(m,n-1)/(m-n+\mu/2+1)\alpha n^2,\ n \geq 1$$

$$B(m,0) = (m+\mu/2)B(m-1,0),\ m \geq 1 \qquad \text{(A-22)}$$

$$B(0,0) = [\Gamma(\mu/2+1)]^2$$

$$C(m,n,k) = \begin{cases} 0, & k \geq 1 \\ 1, & k = 0 \end{cases} \qquad \text{(A-23)}$$

$E\{Z_B^{\mu}\}$--Equation (38)

$$f_2 \equiv (N\alpha)^{\mu}\ \Gamma(\mu+1) \tag{A-24}$$

A(m): same as (A-17)

$$B(m,n) \equiv (\mu+1)_n\ \Gamma(m+\tfrac{1}{2})/\alpha^n(n!)^2\Gamma(m-n+\tfrac{1}{2})$$

$$= (\mu+n)\ (m-n+\tfrac{1}{2})\ B(m,n-1)/\alpha n^2,\ n \geq 1 \tag{A-25}$$

$$B(m,0) = B(0.0) = 1$$

C(m,n,k): same as (A-23)

APPENDIX B

COMPUTATION OF INVERSE PROBABILITY INTEGRAL

An asymptotic expansion for arbitrary probability distributions, adapted from § 26.2.49 of reference (2), is useful for computing minimum detectable signal (MDS). Known as the Cornish-Fisher expansion, it can be written

$$\tau_p \sim m + \sigma W(x_p)/\sqrt{M} \tag{B-1}$$

with

$$\begin{aligned} w(x) = x &+ [\gamma_1 d_2] + [\gamma_2 d_3 + \gamma_1^2 d_4] \\ &+ [\gamma_3 d_5 + \gamma_1 \gamma_2 d_6 + \gamma_1^3 d_7] \\ &+ [\gamma_4 d_8 + \gamma_2^2 d_9 + \gamma_1 \gamma_3 d_{10} + \gamma_1^2 \gamma_2 d_{11} + \gamma_1^4 d_{12}] \\ &+ \ldots \end{aligned} \tag{B-2}$$

where, by definition

$$\Pr\{z > \tau_p\} = p. \tag{B-3}$$

The random variable z, the detector decision variable, is assumed to be the mean of M independent, identically-distributed samples $\{z_i\}$ with moments and cumulants as follows:

$$z = \frac{1}{M} \sum_{i=1}^{M} z_i \tag{B-4}$$

$$m = \kappa_1 = E\{z_i\}$$

$$\sigma_2 = \kappa_2 = \mathrm{Var}\{z_i\} \tag{B-5}$$

$$\kappa_3 = E\{Z_i^3\} - 3\kappa_2\kappa_1 - \kappa_1^3$$

$$\kappa_4 = E\{Z_i^4\} - 4\kappa_3\kappa_1 - 3\kappa_2^2 - 6\kappa_2\kappa_1^2 - \kappa_1^4$$

$$\kappa_5 = E\{Z_i^5\} - 5\kappa_4\kappa_1 - 10\kappa_2\kappa_3 - 10\kappa_3\kappa_1^2 - 15\kappa_1\kappa_2^2 - 10\kappa_2\kappa_1^3 - \kappa_1^5 \tag{B-6}$$

$$\kappa_6 = E\{Z_i^6\} - 6\kappa_5\kappa_1 - 15\kappa_2\kappa_4 - 10\kappa_3^2 - 15\kappa_4\kappa_1^2 - 60\kappa_1\kappa_2\kappa_3$$

$$-15\kappa_2^3 - 20\kappa_3\kappa_1^3 - 45\kappa_1^2\kappa_2^2 - 15\kappa_2\kappa_1^4 - \kappa_1^6$$

$$\gamma_1 = \kappa_3/(M\kappa_2^3)^{\frac{1}{2}}$$

$$\gamma_2 = \kappa_4/M\kappa_2^2$$

$$\gamma_3 = \kappa_5/\ (M^3\kappa_2^5)^{\frac{1}{2}} \tag{B-7}$$

$$\gamma_4 = \kappa_6/M^2\kappa_2^3$$

The number x_p and the coefficients $\{d_i\}$ are related to the Gaussian distribution and are given in Table (2) for several values of p, where columns 2-4 were taken from page 936 of Reference (2).

The brackets around the terms in (B-6) correspond to orders of magnitude with respect to M. A test case run for the non-central chi-squared distribution and using only the first two bracketed terms yielded results 8% and 3% below the true value of MDS for M = 5 and M = 50, respectively.

These coefficients can also be used to approximate the probability density function via the Edgeworth series, as shown in reference (4).

[4] L. E. Miller, "Computing R.O.C. for Quadratic Detectors," NSWC/WOL TR 76-148, 10 Oct 1976.

TABLE 2

COEFFICIENTS FOR CORNISH-FISHER EXPANSION

	P					
	.0001	.001	.01	.1	.5	.9
$d_1 \equiv x_p$	3.71902	3.09022	2.32635	1.28155	0	-1.28155
d_2	2.13852	1.42491	.73532	.10706	-.16667	.10706
d_3	1.67838	.84331	.23379	-.07249	0	.07249
d_4	-2.34115	-1.21025	-.37634	.06106	0	-.06106
d_5	.92761	.30746	-.00152	-.03464	.02500	-.03464
d_6	-5.17267	-1.89355	-.17621	.14644	-.08333	.14644
d_7	4.87514	1.86787	.25195	-.11629	.05247	-.11629
d_8	.35118	.04591	-.03176	.00227	0	-.00227
d_9	-2.62416	-.59060	.07888	.00776	0	-.00776
d_{10}	-3.48080	-.70464	.16058	.01086	0	-.01086
d_{11}	17.56966	4.29304	-.32621	-.10858	0	.10858
d_{12}	-12.61271	-3.32708	.07286	.09585	0	-.09585

APPENDIX C

COMPUTER PROGRAMS

BASIC programs for computing ROC and MDS are listed in the figures which follow. Subroutines common to two or more programs are listed separately. Given the computational outlines of Appendix A, the listings are nearly self-explanatory. Additional comments:

Program QAS. (Figure 6). Given the values of α, τ_{1a}, $K_0(\tau_{1a})$, and $K_1(\tau_{1a})$, computes $Q_A(\tau)$ for the values of h^2 specified by the user.

Program QBS. (Figure 7). Given the values of α and τ_a, computes $Q_B(\tau)$ for the values of h^2 specified by the user.

Program MDA. (Figure 8). Given the values of α, P_{FA}, and WT, computes false alarm threshold via Cornish-Fisher. Then computes detection thresholds for the values of h^2 specified by the user, who interpolates to get MDS.

Program MDB. (Figure 9). Same as MDA but for statistic B.

```
PROGRAM   QAS534

1 DIM G(100),K(110)
2 P1=4*ATN(1)
5 REM        PROGRAM TO COMPUTE
7 PRINT "PROBABILITY INTEGRAL FOR STATISTIC A (WT=1)"
10 READ A1,T1,A
12 DATA 1,5.8,.01
15 PRINT "GIVEN: ALPHA,TAU1,PFA ="A1,T1,A
16 PRINT
17 PRINT "H(DB)","PD","LAST M"
20 GOSUB 700
25 K2=-1
30 H=10↑K2
35 X=H/A1
40 M=0
45 GOSUB 800
50 A2=B2=1
60 C2=1
65 S2=G(0)*K(1)
70 M=M+1
75 K(M+1)=2*M*K(M)/T1+K(M-1)
80 IF M<2 THEN 90
85 GOSUB 860
90 A2=A2*H*T1/2/M↑2
100 B3=B2
105 S1=B2*K(M+1)
110 FOR N=1 TO M
115 B3=B3*2*(M-N+1)*(M-N+.5)/N/A1/T1
120 C2=1
125 S0=K(M-N+1)
130 FOR J=1 TO N
135 C2=C2*T1/2/J
140 IF (M-N-J+1)<0 THEN 155
145 S0=S0+C2*K(M-N-J+1)
150 GOTO 160
155 S0=S0+C2*K(N+J-M-1)
160 NEXT J
165 S1=S1+B3*S0
170 NEXT N
175 L=S2
180 S2=S2+A2*G(M)*S1
185 L=(S2-L)/S2
188 L2=0
190 IF L>.0001 THEN 70
195 P=S2*T1*EXP(-H-X)
196 IF P<.5 THEN 202
197 IF L2=1 THEN 202
198 L2=1
199 L=10*L
200 GOTO 190
202 PRINT 10*K2,P,M
203 IF P<L1 THEN 300
204 L1=P
205 IF P>.99 THEN 300
210 K2=K2+.1
215 IF P>2*A THEN 30
220 K2=K2+.4
225 GOTO 30
300 STOP
700 REM SUBROUTINE FOR BESSEL FUNCTION
705 K(0)=.5101258183*EXP(-5.8)
710 K(1)=.5524676495*EXP(-5.8)
715 RETURN
```

FIGURE 6 PROGRAM TO COMPUTE PROBABILITY INTEGRAL FOR STATISTIC A (WT=1)

```
PROGRAM   QBS534

5 DIM G(100)
10 P1=4*ATN(1)
15 REM        PROGRAM TO COMPUTE
20 PRINT "PROBABILITY INTEGRAL FOR STATISTIC B (WT=1)"
25 READ A1,T,A
30 DATA 1,4.60517,.01
35 PRINT "GIVEN: ALPHA, TAU, PFA "A1,T,A
40 PRINT
45 PRINT "H(DB)","PD","LAST M"
50 PRINT
55 T1=T
60 K2=-2
65 H=10↑K2
70 X=H/A1
75 M=0
80 GOSUB 800
85 A2=B2=1
95 S2=G(0)
100 M=M+1
105 IF M<2 THEN 115
110 GOSUB 860
115 A2=A2*H/M
125 B3=B2
130 S1=B2
135 FOR N=1 TO M
140 B3=B3*(M-N+.5)/A1/N
145 C2=S0=1
150 FOR J=1 TO N
155 C2=C2*T1/J
160 S0=S0+C2
165 NEXT J
170 S1=S1+B3*S0
175 NEXT N
180 L=S2
185 S2=S2+A2*G(M)*S1
190 L=(S2-L)/S2
195 IF L>.0001 THEN 100
200 P=S2*EXP(-T1-H-X)
205 PRINT 10*K2,P,M
207IF P<L1 THEN 300
208 L1=P
210 IF P>.99 THEN 300
215 K2=K2+.1
220 IF P>2*A THEN 65
225 K2=K2+.4
230 GOTO 65
300 STOP
```

FIGURE 7 PROGRAM TO COMPUTE PROBABILITY INTEGRAL FOR STATISTIC B(WT=1)

```
PROGRAM   MDA534

5 DIM G(100),V(6),E(6),K(6)
6 DIM D(12)
10 REM      PROGRAM TO COMPUTE
15 PRINT "FA THRESHOLD AND MDS FOR STATISTIC A"
20 READ A1,A,M1
25 DATA 1,.01,10
30 PRINT "GIVEN: ALPHA, QA, WT ="A1,A,M1
35 PRINT
40 GOSUB 600
45 PRINT "H(DB)","TA","LAST M"
50 PRINT
55 K3=-.2
57 K4=-.11
60 K5=.01
65 FOR K2=K3 TO K4 STEP K5
70 H=10↑K2
75 X=H/A1
80 M9=2
85 GOSUB 800
90 FOR I=1 TO 6
95 G=V(I)
100 M=0
105 S2=G(0)*G↑2
110 A2=1
115 B3=G↑2
120 M=M+1
125 IF M<M9 THEN 135
130 GOSUB 860
135 A2=A2*H/M↑2
140 B3=B3*(M+I/2)
145 S1=B2=B3
150 FOR N=1 TO M
155 B2=B2*(M-N+.5)*(M-N+1)*(N+I/2)/(M-N+I/2+1)/A1/N↑2
160 S1=S1+B2
165 NEXT N
167 L=S2
170 S2=S2+A2*G(M)*S1
175 L=(S2-L)/S2
180 IF L>.0001 THEN 120
185 E(I)=S2*A1↑(I/2)*EXP(-H-X)
190 M9=M
200 NEXT I
205 GOSUB 1100
210 GOSUB 1500
215 PRINT 10*K2,T,M9
220 NEXT K2
300 STOP
600 REM DEFINE GAMMA(1+I/2)=V(I)
605 V(1)=SQR(ATN(1))
610 V(2)=1
615 V(3)=V(1)*3/2
620 V(4)=2
625 V(5)=V(3)*5/2
630 V(6)=6
635 REM    FALSE ALARM THRESHOLD
640 FOR I=1 TO 6
645 E(I)=V(I)↑2*A1↑(I/2)
650 NEXT I
655 GOSUB 1100
660 GOSUB 1500
665 PRINT "FALSE ALARM THRESHOLD ="T
670 PRINT
675 A=.5
680 RETURN
```

FIGURE 8 PROGRAM TO COMPUTE FA THRESHOLD AND MDS FOR STATISTIC A

PROGRAM   MDB534

```
5 DIM G(100),V(6),E(6),K(6)
6 DIM D(12)
10 REM       PROGRAM TO COMPUTE
15 PRINT "FA THRESHOLD AND MDS FOR STATISTIC B"
20 READ A1,A,M1
25 DATA 1,.01,10
30 PRINT "GIVEN:ALPHA, PFA, WT ="A1,A,M1
35 PRINT
40 GOSUB 600
45 PRINT "H(DB)","TA","LAST M"
50 PRINT
55 K3=-.2
57 K4=-.11
60 K5=.01
65 FOR K2=K3 TO K4 STEP K5
70 H=10↑K2
75 X=H/A1
80 M9=2
85 GOSUB 800
90 FOR I=1 TO 6
100 M=0
105 S2=G(0)
110 A2=1
120 M=M+1
125 IF M<M9 THEN 135
130 GOSUB 860
135 A2=A2*H/M
145 S1=B2=1
150 FOR N=1 TO M
155 B2=B2*(I+N)*(M-N+.5)/A1/N↑2
160 S1=S1+B2
165 NEXT N
167 L=S2
170 S2=S2+A2*G(M)*S1
175 L=(S2-L)/S2
180 IF L>.0001 THEN 120
185 E(I)=S2*V(I)*A1↑I*EXP(-H-X)
190 M9=M
200 NEXT I
205 GOSUB 1100
210 GOSUB 1500
215 PRINT 10*K2,T,M9
220 NEXT K2
300 STOP
600 REM DEFINE FACTORIAL
605 V(1)=1
610 V(2)=2
615 V(3)=6
620 V(4)=24
625 V(5)=120
630 V(6)=720
635 REM     FALSE ALARM THRESHOLD
640 FOR I=1 TO 6
645 E(I)=V(I)*A1↑I
650 NEXT I
655 GOSUB 1100
660 GOSUB 1500
665 PRINT "FALSE ALARM THRESHOLD ="T
670 PRINT
675 A=.5
680 RETURN
```

FIGURE 9 PROGRAM TO COMPUTE FA THRESHOLD AND MDS FOR STATISTIC B

```
800 REM SUBROUTINE FOR 1F1(.5;M+1;X)
805 A9=B9=U=V=J9=1
810 A9=A9*X*(J9-.5)/J9↑2
815 B9=B9*X*(J9-.5)/J9/(J9+1)
820 U=U+A9
825 V=V+B9
830 IF A9<1E-13 THEN 845
835 J9=J9+1
840 GOTO 810
845 G(0)=U
850 G(1)=V
855 RETURN
860 G(M)=2*M*((M-1+X)*G(M-1)-(M-1)*G(M-2))/(2*M-1)/X
865 IF G(M)<G(M-1) THEN 875
870 G(M)=1
875 IF G(M)>=1 THEN 885
880 G(M)=1
885 RETURN
1100 REM  COMPUTE CUMULANTS
1105 K(1)=E(1)
1110 K(2)=E(2)-K(1)↑2
1115 K(3)=E(3)-3*K(2)*K(1)-K(1)↑3
1120 K(4)=E(4)-4*K(1)*K(3)-3*K(2)↑2-6*K(2)*K(1)↑2-K(1)↑4
1125 K(5)=E(5)-5*K(1)*K(4)-10*K(2)*K(3)-10*K(3)*K(1)↑2
1130 K(5)=K(5)-15*K(1)*K(2)↑2-10*K(2)*K(1)↑3-K(1)↑5
1135 K(6)=E(6)-6*K(1)*K(5)-15*K(2)*K(4)-10*K(3)↑2-15*K(4)*K(1)↑2
1140 K(6)=K(6)-60*K(1)*K(2)*K(3)-15*K(2)↑3-20*K(3)*K(1)↑3
1145 K(6)=K(6)-45*K(1)↑2*K(2)↑2-15*K(2)*K(1)↑4-K(1)↑6
1148 M2=SQR(M1)
1150 R1=K(3)/M2/K(2)↑1.5
1155 R2=K(4)/M1/K(2)↑2
1160 R3=K(5)/M2↑3/K(2)↑2.5
1165 R4=K(6)/M1↑2/K(2)↑3
1170 RETURN
```

FIGURE 10 SUBROUTINES FOR HYPERGEOMETRIC FUNCTION AND FOR CUMULANTS

```
1500 REM CORNISH FISHER ROUTINE FOR INVERSE PROBABILITY INTEG.
1505 REM ADMISSIBLE PROB: .001,.01,.5
1510 IF A<.009 THEN 1550
1515 IF A>.1 THEN 1580
1520 D(1)=2.32635
1522 D(2)=.73532
1524 D(3)=.23379
1525 D(4)=-.37634
1526 D(5)=-.00152
1528 D(6)=-.17621
1530 D(7)=.25195
1532 D(8)=-.03176
1534 D(9)=.07888
1535 D(10)=.16058
1536 D(11)=-.32621
1540 D(12)=.07286
1545 GOTO 1605
1550 D(1)=3.09022
1552 D(2)=1.42491
1554 D(3)=.84331
1555 D(4)=-1.21025
1556 D(5)=.30746
1558 D(6)=-1.89355
1560 D(7)=1.86787
1562 D(8)=.04591
1564 D(9)=-.59060
1565 D(10)=-.70464
1567 D(11)=4.29304
1570 D(12)=-3.32708
1575 GOTO 1605
1580 D(1)=D(3)=D(4)=0
1582 D(2)=-.16667
1584 D(5)=.025
1585 D(6)=-.08333
1587 D(7)=.05247
1590 D(8)=D(9)=D(10)=0
1595 D(11)=D(12)=0
1605 M2=SQR(M1)
1610 W=D(1)+R1*D(2)   +(R2*D(3)+R1↑2*D(4))     +R3*D(5)
1615 W=W+(R1*R2*D(6)+R1↑3*D(7))     +(R4*D(8)+R2↑2*D(9))
1620 W=W+(R1*R3*D(10)+R1↑2*R2*D(11)+R1↑4*D(12))
1625 T=E(1)+SQR(K(2))*W/M2
1630 RETURN
```

FIGURE 11 SUBROUTINE FOR CORNISH – FISHER EXPANSION